Group classification of the general evolution equation: local and quasilocal symmetries

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Abstract

We give a review of our recent results on group classification of the most general nonlinear evolution equation in one spatial variable. The method applied relies heavily on the results of our paper *Acta Appl. Math.*, **69**, 2001, in which we obtain the complete solution of group classification problem for general quasilinear evolution equation.

In this paper we briefly review our recent results on group classification of the general nonlinear evolution equation

$$u_t = F(t, x, u, u_x, u_{xx}). \tag{1}$$

Here $u = u(t, x)$, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$; F is an arbitrary smooth function obeying the restriction $\partial F/\partial u_{xx} \neq 0$.

Using the standard Lie approach we prove that the maximal invariance group of equation (1) is generated by the operator

$$\mathrm{Q} = \tau(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \tag{2}$$

where functions τ, ξ and η are arbitrary solutions of a single partial differential equation (PDE)

$$\begin{aligned}\eta_t - u_x\xi_t + (\eta_u - \tau_t - u_x\xi_u)F = \Big(\eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_u\Big)F_{u_x}\\ +\Big(\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_x^2(\eta_{uu} - 2\xi_{xu}) - u_x^3\xi_{uu}\\ +u_{xx}(\eta_u - 2\xi_x) - 3u_x u_{xx}\xi_u\Big)F_{u_{xx}} + \tau F_t + \xi F_x + \eta F_u.\end{aligned} \tag{3}$$

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So to obtain (exhaustive) group classification of the class of equations (1) we need to construct *all* possible functions, τ, ξ, η and F, obeying the above constraint (determining equation). Evidently the challenge of the problem is in the word *all*, since the system of classifying equations is not over-determined (as is customary for this type of problems). Moreover it is under-determined. This is the reason why the numerous papers devoted to group classification of nonlinear evolution equations deal mostly with classes of PDEs depending on arbitrary functions of one, or at most two, variables.

A starting point of our analysis is a simple observation that solutions $\mathbf{v}_a = (\tau_a, \xi_a, \eta_a),\ a = 1, \ldots, n$, of (3) span a Lie algebra ℓ. So without any loss of generality we can replace (3) with the (possibly infinite) set of systems of PDEs

$$\begin{cases} \text{Equation(3)}, \\ [Q_i,\ Q_j] = C_{ij}^k\, Q_k, \end{cases} \tag{4}$$

or, equivalently,

$$\begin{cases} \text{Equation(3)}, \\ Q_i\tau_j - Q_j\tau_i = C_{ij}^k\, \tau_k, \\ Q_i\xi_j - Q_j\xi_i = C_{ij}^k\, \xi_k, \\ Q_i\eta_j - Q_j\eta_i = C_{ij}^k\, \eta_k. \end{cases} \tag{5}$$

In the above formulas the indices i, j, k take the values $1, \ldots, n$ ($n \geq 1$ is a dimension of the corresponding Lie algebra), and C_{ij}^k are structure constants of the Lie algebra ℓ.

If we solve the (over-determined) system of PDEs (5) for *all* possible dimensions $n \geq 1$ of *all* admissible Lie algebras, ℓ, then the problem of group classification of Eq.(1) is completely solved. In other words the problem of group classification of the general evolution equation (1) reduces to integrating over-determined systems of PDEs (5) for all $n = 1, 2, \ldots, n_0$, where n_0 is the maximal dimension of the Lie algebra admitted by the equation under study.

One way to handle the above problem would be starting with investigating compatibility of systems (5) for all $n \geq 1$. This strategy is close in spirit to Reid's procedure of describing the algebra admitted by PDE without integrating determining equations [1].

However, a more natural approach is actually to integrate equations (5) so that compatibility conditions come as a by-product. This is even more so if we take into account that low-dimensional abstract Lie algebras are described up to the dimension $n = 6$ (mainly due to efforts by Mubarakzyanov [2, 3]). So, if we

1. construct all realizations of Lie algebras by operators, the coefficients of which satisfy Eq.(3), up to some fixed dimension n_0, and
2. prove that (1) does not admit invariance algebras of the dimension $n > n_0$,

then the problem of group classification of Eq.(1) is completely solved.

The underlying ideas of the above approach are rather natural. No wonder that they have already been used in various contexts. In particular Fushchych & Serov [4] exploited them to classify conformally-invariant wave equations in the multidimensional case. In a more systematic way these ideas have been utilized by Gagnon & Winternitz [5] to classify variable coefficient Schrödinger equations.

In its present form the approach formulated above has been developed in [6], where we perform preliminary group classification of nonlinear Schrödinger equations. Later we applied this approach to classify second quasilinear evolution equations [7, 8], third-order evolution equations [11] and nonlinear wave equations [12] in one spatial variable.

We perform group classification within the action of equivalence group preserving the class of PDEs under study. It is not difficult to prove that the maximal equivalence transformation group preserving class (1) is

$$\bar{t} = T(t), \quad \bar{x} = X(t,x,u), \quad \bar{u} = U(t,x,u), \tag{6}$$

where

$$T' = \frac{dT}{dt} \neq 0, \quad \frac{D(X,U)}{D(x,u)} \neq 0. \tag{7}$$

In the paper [7] we obtain an exhaustive group classification of (quasilinear) evolution equation

$$v_t = f(t,x,v,v_x)v_{xx} + g(t,x,v,v_x), \quad v = v(t,x). \tag{8}$$

Those results provide almost complete solution of the problem of group classification for general evolution equation (1). This claim follows from the fact that, if Eq.(1) admits a one-parameter group with the infinitesimal generator

$$Q = \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \tag{9}$$

then it is transformed into an equation of the form (1).

Indeed operator (9) can be reduced to the canonical form $\partial_{u'}$ by a suitable change of variables

$$t' = T(t), \quad x' = X(t,x,u), \quad u' = U(t,x,u) \tag{10}$$

(note that the above transformation belongs to the equivalence group of equation (1)). The corresponding invariant equation takes the form

$$u'_{t'} = F'(t', x', u'_{x'}, u'_{x'x'}).$$

Differentiating the obtained equation by x', replacing $u'_{x'}$ with $v(t',x')$ and dropping the primes we arrive at the quasilinear PDE of the form (8).

Before proceeding to exploit the above fact any further, we briefly summarize the principal results of [7].

- There are 2 inequivalent PDEs (8) admitting one-dimensional algebras;

- There are 5 inequivalent PDEs (8) admitting two-dimensional algebras.
- There are 34 inequivalent PDEs (8) admitting three-dimensional algebras.
- There are 35 inequivalent PDEs (8) admitting four-dimensional algebras.
- There are 6 inequivalent PDEs (8) admitting five-dimensional algebras.

As an example, we give below the complete list of inequivalent equations (8) invariant under five-dimensional Lie algebras.

$$
\begin{aligned}
&u_t = u^{-4}\, u_{xx} - 2u^{-5}\, u_x^2,\\
&u_t = u_{xx} + x^{-1}\, u\, u_x - x^{-2}\, u^2 - 2x^{-2}\, u,\\
&u_t = u_x^{-2}\, u_{xx} + u_x^{-1},\\
&u_t = \mathrm{e}^{u_x}\, u_{xx},\\
&u_t = u_x^n\, u_{xx}, \quad n \geq -1,\ n \neq 0,\\
&u_t = (1+u_x^2)^{-1}\, \exp(n \arctan u_x)\, u_{xx}.
\end{aligned}
$$

It follows from the above considerations that, if the invariance algebra of equation (1) contains an operator of the form (9), then it is equivalent to equation (8) the group properties of which are already known so that to complete group classification of Eq.(1) we need to describe all equations (1) the invariance algebras of which are spanned by the operators

$$
Q_i = \tau_i(t)\partial_t + \xi_i(t,x,u)\partial_x + \eta_i(t,x,u)\partial_u, \quad i = 1, \ldots, n, \tag{11}
$$

where the functions $\tau_1(t), \ldots, \tau_n(t)$ are linearly independent. We denote the class of such equations as $\mathcal{L}_1$.

We prove that the highest dimension of an invariance algebra of Eq.(8) belonging to $\mathcal{L}_1$ equals 3. The algebra in question is $sl(2,\mathbf{R})$ and the corresponding invariant equations are given below.

$$
\begin{aligned}
u_t &= x^{-1}\, u\, u_x - x^{-2}\, u^2 + x^{-2}\, \tilde{F}(x^2 u_{xx} - 2u,\ 2u - x u_x),\\
sl(2,\mathbf{R}) &= \left\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t - tx\partial_x + x^2\partial_u,\ \partial_t \right\rangle;\\
u_t &= -\frac{1}{4}\, x^{-1}\, u_x + x^{-3}\, u_x^{-1}\, \tilde{F}(u,\ u_x^{-2} u_{xx} + 3x^{-1} u_x^{-1});\\
sl(2,\mathbf{R}) &= \left\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t + x(x^2 - t)\partial_x,\ \partial_t \right\rangle.
\end{aligned}
$$

Here $\tilde{F}$ is an arbitrary smooth function.

There are only two equations from $\mathcal{L}_1$ admitting lower-dimensional invariance algebras, namely,

$$
\begin{aligned}
&u_t = \tilde{F}(x,\ u^{-1}u_x,\ u^{-1}u_{xx}), \quad \ell = \langle -t\partial_t - u\partial_u,\ \partial_t \rangle.\\
&u_t = \tilde{F}(x,\ u,\ u_x,\ u_{xx}), \quad \ell = \langle \partial_t \rangle.
\end{aligned}
$$

In the above formulas $\tilde{F}$ is an arbitrary smooth function.

Equations from $\mathcal{L}_1$ together with invariant equations of the form (8) provide the complete solution of the problem of classifying equations (1) that admit nontrivial Lie symmetry.

As we noted in [8], results of the group classification of (1) can be utilized to derive their quasilocal symmetries. The term quasi-local has been introduced independently in [9] and [10] to distinguish nonlocal symmetries that are equivalent to local ones through nonlocal transformation.

We have already shown that equations (1) and (8) are related through the nonpoint transformation $v(t,x) = u_x(t,x)$ or, inversely, $u(t,x) = \partial_x^{-1} v(t,x)$. Suppose now that Eq.(1) admits the one-parameter transformation group

$$\begin{cases} t' = T(t,\theta), \\ x' = X(t,x,u,\theta), \\ u' = U(t,x,u,\theta). \end{cases}$$

Computing the first prolongation of the above formulas gives the transformation rule for the first derivative of u'

$$\frac{\partial u'}{\partial x'} = \frac{u_x U_u + U_x}{u_x X_u + X_x}.$$

In terms of the variables, $t, x, v(t,x)$, this transformation group is

$$\begin{cases} t' = T(t,\ \theta), \\ x' = X(t,\ x,\ u,\ \theta), \\ v' = \dfrac{vU_u(t,\ x,\ u(t,\ x),\ \theta) + U_x(t,\ x,\ u(t,\ x),\ \theta)}{vX_u(t,\ x, u(t,\ x),\ \theta) + X_x(t,\ x,\ u(t,\ x),\ \theta)}, \end{cases}$$

where $u(t,x) = \partial^{-1} v(t,x)$. Consequently, if the relation

$$X_{uu}^2 + X_{xu}^2 + U_{uu}^2 + U_{ux}^2 \neq 0 \tag{12}$$

holds, the transformed equation (8) possesses a quasilocal symmetry. If a symmetry group of Eq.(1) satisfies constraint (12), then we say that this equation belongs to the class $\mathcal{L}_2$. In what follows we describe all equations from $\mathcal{L}_2$, the symmetry algebras of which are at most three-dimensional.

It is not difficult to become convinced of the fact that the class $\mathcal{L}_2$ does not contain equations the maximal invariance algebras of which are of the dimension $n \leq 2$. Below we give the full list of inequivalent equations belonging to $\mathcal{L}_2$ and admitting three-dimensional Lie algebras (we follow notations of [8])

Algebra $sl(2,\mathbf{R})$:

1. Realization

   $Q_1 = \partial_u, \qquad Q_2 = 2u\partial_u - x\partial_x, \qquad Q_3 = -u^2\partial_u + xu\partial_x.$

   Invariant equation :

   $u_t = xu_x\, \tilde{F}(t,\ x^{-5}\, u_x^{-3}\, u_{xx} + 2x^{-6}\, u_x^{-2}).$

2. Realization

   $Q_1 = \partial_u, \qquad Q_2 = 2u\partial_u - x\partial_x.$

$Q_3 = (\varepsilon x^{-4} - u^2)\partial_u + xu\partial_x,\ \varepsilon = \pm 1.$

Invariant equation :

$$u_t = x^{-2}\sqrt{x^6\, u_x^2 + 4\varepsilon}\, \tilde{F}\left(t,\ (x^6\, u_x^2 + 4\varepsilon)^{-\frac{3}{2}}\,(x^4\, u_{xx} + 5x^3\, u_x + \frac{1}{2}\, x^9\, u_x^3)\right).$$

Algebra $so(3)$:

1. Realization

$Q_1 = \partial_u, \qquad Q_2 = \cos u\partial_x + \tan x \sin u\partial_u,$

$Q_3 = -\sin u\partial_x + \tan x \cos u\partial_u.$

Invariant equation :

$$u_t = \sqrt{\sec^2 x + u_x^2}\, \tilde{F}\left(t,\ \left(u_{xx}\, \cos x - (2 + u_x^2\, \cos^2 x)\, u_x\, \sin x\right) \times (1 + u_x^2\, \cos^2 x)^{-\frac{3}{2}}\right).$$

Algebra $A_{3.8}$:

1. Realization

$Q_1 = \partial_u, \qquad Q_2 = x\partial_u, \qquad Q_3 = -(x^2+1)\partial_x - xu\partial_u.$

Invariant equation

$$u_t = \sqrt{1+x^2}\, \tilde{F}\left(t,\ u_{xx}\,(1+x^2)^{\frac{3}{2}}\right).$$

2. Realization

$Q_1 = \partial_u, \qquad Q_2 = -\tan(t+x)\partial_u, \qquad Q_3 = \partial_t + \tan(t+x)u\partial_u.$

Invariant equation

$$u_t = u_x + \sec(t+x)\, F\left(x,\ u_{xx}\, \cos(t+x) - 2u_x\, \sin(t+x)\right).$$

Algebra $A_{3.9}$:

1. Realization

$Q_1 = \partial_u, \qquad Q_2 = x\partial_u,$

$Q_3 = -(x^2+1)\partial_x + (q-x)u\partial_u,\ q \neq 0.$

Invariant equation

$$u_t = \mathrm{e}^{-q\arctan x}\sqrt{1+x^2}\, \tilde{F}\left(t,\ u_{xx}\, \mathrm{e}^{q\arctan x}\,(1+x^2)^{\frac{3}{2}}\right).$$

2. Realization

$Q_1 = \partial_u, \qquad Q_2 = -\tan(t+x)\partial_u,$

$Q_3 = \partial_t + (q + \tan(t+x))u\partial_u,\ q \neq 0.$

Invariant equation

$$u_t = u_x + \sec(t+x)\, \mathrm{e}^{qt}\, \tilde{F}\left(x,\ \mathrm{e}^{-qt}\,(u_{xx}\, \cos(t+x) - 2u_x\, \sin(t+x))\right).$$

Differentiating any of the above equations by x and replacing u_x with v yields an equation of the form (8) that admits a quasilocal symmetry. Consider, as an example, equation

$$u_t = u_x + \sec(t+x)\, F\left(x,\ u_{xx}\, \cos(t+x) - 2u_x\, \sin(t+x)\right)$$

which is invariant with respect to the algebra $\langle \partial_t + \tan(t+x)u\partial_u \rangle$. The corre-

sponding one-parameter transformation group is

$$\begin{cases} t' = t, \\ x' = x, \\ u' = u\,\sec(t + x + \theta), \end{cases}$$

where $\theta \in \mathbf{R}$ is the group parameter. Computation of the first prolongation of the above formulas yields

$$u'_{x'} = u_x\,\sec(t + x + \theta) + u\,\sec(t + x + \theta)\,\tan(t + x + \theta)$$

or

$$v'_{x'}(t',\,x') = \Big(v(t,\,x) + u(t,\,x)\,\tan(t + x + \theta)\Big)\sec(t + x + \theta),$$

where $u(t, x) = \partial^{-1} v(t, x)$. The corresponding equation for $v = v(t, x)$ is

$$v_t = v_x + \sec(t + x)\,\left(\tan(t + x)\,\tilde{F} + \tilde{F}_{\omega_1}\right) + \left(v_{xx} - 3\tan(t + x)\,v_x - 2v\right)\,\tilde{F}_{\omega_2}.$$

Here $\tilde{F}$ is an arbitrary smooth function of the variables

$$\omega_1 = x \text{ and } \omega_2 = \cos(t + x)\,v_x - 2\sin(t + x)\,v.$$

1 Concluding Remarks

The motivation for writing this paper is to introduce our strategy for attacking the problem of group classification of the most general evolution equation in one spatial dimension. The three basic elements of our analysis are

- Group classification of the partial case of the class of PDEs in question, of the quasilinear evolution equations.
- Description of nonlinear PDEs, which belong to the class $\mathcal{L}_1$.
- Description of nonlinear PDEs, which belong to the class $\mathcal{L}_2$.

It is one of the principal results of the present paper that the complete solution of these three subproblems yields a final solution of the problem of group classification of the class of PDEs (1).

While the first two subproblems have already been solved (see, citeren1,ren2 and this paper), the classification of $\mathcal{L}_2$ evolution equations is still to be completed. We present the classification results for the Lie algebras which have dimension not higher than three. The work on higher dimensional Lie algebras is in progress now and will be reported elsewhere.

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